Spatial Interactions

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Abstract: The social sciences regularly postulate that the behavior of individuals and groups is influenced by their degree of interaction with each other. But while scholars use such information to create **W** in spatial regression models, their interpretation as merely "weights" leads scholars to dismiss their predictive power outside of the creation of spatial lags. I argue that scholars' view of **W** needs to shift from a set of spatial weights to a set of conditioning variables; specifically, by introducing a framework of spatial regression models as multiplicative interaction models with some unique-and sometimes problematic-assumptions about the coefficients. Because the row-sum of **W** is not included as a covariate when estimating spatial regression models, a predictable form of omitted variable bias affects the coefficients of spatial lags and their subsequent marginal effects. But by recognizing these models as multiplicative interaction models, scholars can use marginal effects plots to clarify the interpretation of these models.

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Social science focuses on how the behavior of a human(s) is influenced by its interaction with other humans. Indeed, political science regularly studies how the behavior of individuals and groups is influenced by their degree of interaction with others in a network. Countries that have more neighbors are more involved in conflicts and alliances (Starr and Most 1976). Individuals with larger social networks have different patterns of political participation (Leighley 1990). Countries within the American network of foreign policy are more democratic (Brinks and Coppedge 2006). The list goes on.

Scholars do not limit their use of associative information between observations to simple additive effects. This information is also used in spatial regression models to construct the spatial weights matrix **W**, a nxn matrix that contains the set of spatial relationships between observations in the data. By post-multiplying **W** with other variables, scholars create spatial lags of variables that are then used to test hypotheses of spatial dependence across observations. Spatial regression models are useful in that they allow scholars to test theories that could only be previously described: countries are more likely to go to war if their neighbors are at war, individuals are more likely to participate politically if their peers are also participating, etc.

But while spatial regression models enable scholars to estimate more nuanced models of social interaction, they normally fail to consider that a unit's degree of interaction with other units may have an effect on behavior beyond what is contained in the spatial lags. This is largely due to the view that the information contained in **W** is merely a weight and, beyond its role in spatial lags, will not have an effect on the behavior of individuals or groups. This approach is inconsistent other political science scholarship which view the information contained in **W** as a variable that if not included by itself within a model can bias the resulting estimates.

I argue that scholars' view of **W** needs to shift from a set of spatial weights to a set of conditioning variables within the model. This is done by introducing a new framework for understanding spatial regression models as multiplicative interaction models with some unique-and sometimes problematic-assumptions about the coefficients. Specifically, a spatially lagged term in a regression model contains n-1 multiplicative interactions without the constitutive terms. When the constitutive terms that represent the a unit's degree of interaction with other units is omitted from the model, a predictable form of omitted variable bias occurs that will bias the coefficients of spatial lags and their subsequent marginal effects. But by recognizing these spatial regression models as multiplicative interaction models and including their constitutive terms, scholars can use common interpretation tools in ways that will greatly improve scholarly understanding of spatial dependence (Brambor, Clark, and Golder 2006).

1 Spatial Regression Models

Spatial regression models account for dependencies among observations in space, broadly defined. Like their time-series counterparts, these models can address dependencies in the dependent variable using a spatial autoregressive model (SAR), in the independent variable(s) using spatial lags of **X** (SLX), and in the disturbance term using a spatial error model (SEM). But time series models are relatively easy to specify, given the universally agreed upon measurement units of time and its unidimensional and unidirectional nature. Spatial regression models, in contrast, are more difficult: the differing units of space complicate measurement, the multidimensionality of space complicates complete model specification, and the multidirectionality of space complicates model estimation.

Some of these problems are solved by the **W** matrix. **W** is a nxn matrix with zeroes along the diagonal that, in the purely cross-sectional context, contains all of the n(n-1) relationships in the data in the off-diagonal elements: each element w_{ij} represents the relative influence unit j has on unit i. More extreme values of w_{ij} indicate a stronger influence of unit j on unit i. Likewise, a 0 value in w_{ij} indicates unit j has no influence on unit i.

For many scholars, Tobler's first law of geography drives the specification of **W** : "everything is related to everything else, but near things are more related than distant things" (1970). Thus, scholars assign larger numbers to observations that are closer together in space. But rarely do scholars recognize that Tobler's first law is a conditional hypothesis. Tobler does not state that all observations in a particular geographic space are dependent upon one another; this would imply a **W** matrix that contains only ones on the non-diagonals. Rather, Tobler states that all observations are dependent upon one another conditional upon the distance between pairs of observations. Scholars have since applied spatial models beyond geographic space (Beck, Gleditsch, and Beardsley 2006). But the conditional hypotheses surrounding their use remains: an observation's value of a dependent variable is, in part, determined by other observations' value of a variable(s), conditional upon the spatial relationship between the two observations.

Such conditional relationships are usually modelled using multiplicative interactions (Brambor, Clark, and Golder 2006). But rarely do scholars approach spatial regression models from a multiplicative interactions framework. This can be attributed to the discussion of **W** as a "weights" matrix in the literature, where a unit's degree of interaction with other units only affects the dependent variable through spatial lags. But the above references show that the degree of interaction between units in space may have an effect unrelated to spatial dependence altogether. Indeed, the growth of network analysis in the social science has allowed for more sophisticated measures of a unit's degree of connectivity to another units in a sample which, in turn, are useful predictors of other important social phenomena (Hafner-Burton, Kahler, and Montgomery 2009, Ward, Stovel, and Sacks 2011). Thus, the current view of the **W** matrix seems erroneous.

Neumayer and Plumper (2012) provide a notable exception. They argue that "most theories of spatial policy dependence either are already inherently conditional or, if not, should be, whereas empirical models with few exceptions estimate an unconditional spatial effect." While they make a number of notable contributions, the most important one for present purposes is their view of a spatial lag as a multiplicative interaction term between a row-standardized spatial lag and the row-sum of the spatial weights matrix. Thus a spatial autoregressive model without row-standardization is a special case of a more general model:

$$y_i = \alpha + \rho_1 \sum_{j \neq i} \left[\frac{w_{ij}}{\sum_{j \neq i} w_{ij}} y_j \right] + \rho_2 \sum_{j \neq i} \left[\frac{w_{ij}}{\sum_{j \neq i} w_{ij}} y_j \right] * \sum_{j \neq i} w_{ij} + \rho_3 \sum_{j \neq i} w_{ij} + \beta x_i + \epsilon_i \quad (1)$$

$$\Leftrightarrow \boldsymbol{y} = \boldsymbol{\iota}\alpha + \boldsymbol{W}^{Row}\boldsymbol{y}\rho_1 + \boldsymbol{W}\boldsymbol{y}\rho_2 + \boldsymbol{W}\boldsymbol{\iota}\rho_3 + \boldsymbol{x}\beta + \boldsymbol{\epsilon} \quad (2)$$

where $\boldsymbol{\iota}$ is a vector of ones and $\boldsymbol{W}^{Row}$ is the row-standardization of $\boldsymbol{W}$.

Despite the recognition of spatial regression models as multiplicative interaction models, however, Neumayer and Plumper do not advocate for best practices when discussing their application (Brambor, Clark, and Golder 2006). The authors claim that $\boldsymbol{W}\boldsymbol{\iota}$ can be omitted when a scholar thinks that "there is no independent effect of the row sum of weights on the dependent variable." Yet viewing $\boldsymbol{W}\boldsymbol{\iota}$ as an independent effect incorrectly implies that its coefficient can be interpreted in the same way as it can in an additive linear model. Rather, ρ_3 is the effect of $\boldsymbol{W}\boldsymbol{\iota}$ when $\boldsymbol{W}\boldsymbol{y}$ is equal to zero. Additionally, the authors state that "in general it will be better to free the coefficients ρ_1 and ρ_3 and to estimate [equation 2 rather than a more restricted model]." While true, the authors do not caution readers that omitting ρ_3 (or ρ_1) when they do not equal zero will result in bias in any estimate of ρ_2.

2 Spatial Interactions

I argue that the conditional relationships modelled using spatial regression are inherently multiplicative interaction models with some restrictive assumptions. This is true no matter the particular spatial model estimated, although the details of the assumptions made when estimating each differ slightly in terms of implications and enormously in terms of mathematical proof. I will demonstrate the logic of this worldview using the simplest spatial model possible, the SLX model with a single independent variable, leaving additional derivations to the reader. Consider the following model:

$$\boldsymbol{y} = \boldsymbol{\iota}\alpha + \boldsymbol{x}\beta + \boldsymbol{W}\boldsymbol{x}\boldsymbol{\theta} + \boldsymbol{\varepsilon} \tag{3}$$

To better see how spatial interactions work in spatial models, I express this model in expanded scalar notation:

$$y_i = \alpha + x_i\beta + \sum_{j \neq i} w_{ij}x_j\theta + \varepsilon_i \tag{4}$$

$$\Leftrightarrow y_i = \alpha + x_i\beta + w_{i1}x_1\theta + w_{i2}x_2\theta + ... + w_{in}x_n\theta + \varepsilon_i \tag{5}$$

I offer a view of spatial regression models that complements existing understanding, but relies on two key observations about the elements of $\mathbf{W}$. First, each x_j in the model is a variable. In a sample of cross-sectional data, each x_j is an observed value of variable $\mathbf{x}$ and is thus a constant. From a frequentist perspective, however, each x_j can also be thought of as an observation from a larger superpopulation of counterfactuals and is thus a variable. Extending beyond a single cross-section, each x_j also represents a variable in it of itself. For many data-generating processes, this can be as part of a time-series x_{jt} or larger cross-section x_{js}.

Second, each w_{ij} is also a variable. Some scholars may be initially skeptical of this view, seeing each w_{ij} as indexing the spatial relationships within the model. But a variable is simply a function that maps all possible outcomes of some event onto the real number line. Each column of the spatial weights matrix w_{ij} meets this definition, where the event is the possible spatial relationships some observation j can have with other observations and the elements of column w_j contain the numerical representations of observed events.

With these two observations explicitly made, my larger view of spatial regression models is that each spatial component in the model contains n-1 multiplicative interactions, in which the effect of each x_j on y_i is conditioned by the spatial relationship between the two observations (as defined by w_{ij}).[1] The interactions are subject to the linear restriction that they all have the same coefficient, θ. This restriction, combined with the assumption that each w_{ij} accurately reflects the spatial relationships in the data, helps scholars confront the inherent identification problem within spatial models that the data contains n(n-1) potential spatial relationships while providing only n observations (McMillen 2012, Halleck Vega and Elhorst 2015).

But while spatial models are inherently multiplicative interactions models, they omit the constitutive terms w_{ij} and x_j from the model. This functionally constrains the effect of these variables to zero. This is a dangerous constraint, as bias will permeate the interaction terms $w_{ij}x_j$ if this constraint is incorrect (Brambor, Clark, and Golder 2006). For the simple SLX model being examined, the true spatial model being estimated is:

$$y_i = \alpha + x_i\beta + w_{i1}0 + x_1 0 + w_{i1}x_1\theta + w_{i2}0 + x_2 0 + w_{i2}x_2\theta + ... + w_{in}0 + x_n 0 + w_{in}x_n\theta + \varepsilon_i \quad (6)$$

[1]Spatial regression models do not contain the interaction $w_{ii}x_i$ by design, as the spatial relationship between an observation and itself is infinite in comparison to other spatial relationships. Instead, the relationship between x_i and y_i is captured with β, and all elements w_{ii} are forced to be zero.

Of course, omitting variables from a model does not necessarily lead to bias. Examining whether the constitutive terms are correlated with the dependent variable and the interaction terms will be the subject of the next sections. But assume for the moment that a scholar wanted to include these constitutive terms in the model. To do so, the scholar could extend the identification strategy for the spatial interaction terms to the constitutive terms by subjecting them the linear restriction that each set of terms are equal. For the SLX model being examined, group the like terms together and factor them:

$$y_i = \alpha + x_i\beta + w_{i1}0 + w_{i2}0 + ... + w_{in}0 + x_1 0 + x_2 0 + ... + x_n 0 + w_{i1}x_1\theta + w_{i2}x_2\theta + ... + w_{in}x_n\theta + \varepsilon_i \quad (7)$$

$$\Leftrightarrow y_i = \alpha + x_i\beta + \sum_{j\neq i} w_{ij}0 + \sum_{j\neq i} x_j 0 + \sum_{j\neq i} w_{ij}x_j\theta + \varepsilon_i \quad (8)$$

and then instead of constraining the two variables to be zero, allow them to have their own coefficient:

$$y_i = \alpha + x_i\beta + \sum_{j\neq i} w_{ij}\theta_1 + \sum_{j\neq i} x_j\theta_2 + \sum_{j\neq i} w_{ij}x_j\theta_3 + \varepsilon_i \quad (9)$$

$$\Leftrightarrow \boldsymbol{y} = \boldsymbol{\iota}\alpha + \boldsymbol{x}\beta + \boldsymbol{W}\boldsymbol{\iota}\theta_1 + \boldsymbol{K}\boldsymbol{x}\theta_2 + \boldsymbol{W}\boldsymbol{x}\theta_3 + \boldsymbol{\varepsilon} \quad (10)$$

where **K** is a hollow matrix and all of its non-diagonal elements are equal to 1.

3 The Necessity of Including the Row-Sum of W as a Variable

In order to determine whether scholars can safely omit these constitutive terms, one must determine their substantive meaning. I do this with each set of terms in turn. In the general

SLX model in equations 9 and 10, the coefficient of $\mathbf{Kx}$ can be interpreted as the effect of x_j on y_i when w_{ij} equals zero. As stated earlier, scholars specify a given w_{ij} to equal zero when they expect x_j to not have an effect on y_i. Thus, the omission of $\mathbf{Kx}$ from the model seems to be a benign one when $\mathbf{W}$ is correctly specified since there is a theoretical reason to expect its coefficient to equal zero.

Indeed, Vande Kamp (n.d.) shows that $\mathbf{Kx}$ functions as a specification test of $\mathbf{W}$. Deriving his conclusions from a measurement error perspective, he shows that the coefficient of $\mathbf{Kx}$ will be statistically indistinguishable from zero when $\mathbf{W}$ is correctly specified but will be statistically distinguishable from zero when the non-diagonal elements of $\mathbf{W}$ are inflated by a constant. While this "K test" is still the subject of ongoing research, I direct interested readers to that manuscript. But for present purposes, $\mathbf{Kx}$ can be safely omitted from the model if $\mathbf{W}$ is correctly specified.

The coefficient of $\boldsymbol{W\iota}$ can be interpreted as the effect of w_{ij} on y_i when x_j equals zero. As previously discussed, there are a number of theoretical reasons to believe that a unit's interaction with other units of analysis might affect a dependent variable of interest. Methodologically, there are also reasons to include $\boldsymbol{W\iota}$ in the model. Suppose that a scholar wants to estimate the SLX model from equation 10 but cannot observe $\mathbf{x}$. Rather, the scholar observes $\mathbf{x*}$ which is the variable $\mathbf{x}$ inflated by a constant c:

$$\boldsymbol{x}* = \boldsymbol{x} + \boldsymbol{\iota} c \tag{11}$$

If the scholar attempts to estimate the model, it will result in the following endogeneity:

$$\boldsymbol{y} = \boldsymbol{\iota}\alpha + (\boldsymbol{x} + \boldsymbol{\iota}c)\beta + \boldsymbol{W}(\boldsymbol{x} + \boldsymbol{\iota}c)\theta + \boldsymbol{\varepsilon} \tag{12}$$

$$\Leftrightarrow \boldsymbol{y} = \boldsymbol{\iota}\alpha + \boldsymbol{x}\beta + \boldsymbol{\iota}c\beta + \boldsymbol{W}\boldsymbol{x}\theta + \boldsymbol{W}\boldsymbol{\iota}c\theta + \boldsymbol{\varepsilon} \tag{13}$$

$$\Leftrightarrow \boldsymbol{y} = \boldsymbol{\iota}(\alpha + c\beta) + \boldsymbol{x}\beta + \boldsymbol{W}\boldsymbol{x}\theta + (\boldsymbol{\varepsilon} + \boldsymbol{W}\boldsymbol{\iota}c\theta) \tag{14}$$

Here, we see that part of the endogeneity from using a measure of $\mathbf{x}$ inflated by a constant is captured in the intercept term α. But an additionaly endogeneity, $\boldsymbol{W\iota}c\theta$ will be stuck in the error term that will almost certainly be correlated with the spatial lag. If, however, $\boldsymbol{W\iota}$ were also included in the regression model, then the endogeneity would be transferred to its coefficient:

$$\boldsymbol{y} = \boldsymbol{\iota}(\alpha + c\beta) + \boldsymbol{x}\beta + \boldsymbol{W}\boldsymbol{\iota}\theta_1 + \boldsymbol{W}\boldsymbol{x}\theta_3 + (\boldsymbol{\varepsilon} + \boldsymbol{W}\boldsymbol{\iota}c\theta) \tag{15}$$

$$\Leftrightarrow \boldsymbol{y} = \boldsymbol{\iota}(\alpha + c\beta) + \boldsymbol{x}\beta + \boldsymbol{W}\boldsymbol{\iota}(\theta_1 + c\theta_3) + \boldsymbol{W}\boldsymbol{x}\theta_3 + \boldsymbol{\varepsilon} \tag{16}$$

Indeed, this proof mirrors the one given by Brambor, Clark, and Golder (2006) when they advocated that constitutive terms should be included in regression models even when there is a theoretical reason to believe they are zero.

While there are strong theoretical and methodological justifications for including $\boldsymbol{W\iota}$ in spatial regression models, there are also a few mathematical considerations one must address. Many scholars row-normalize their $\mathbf{W}$ matrices for theoretical and statistical purposes (though see Neumayer and Plumper 2016). In these matrices, the row-sum will equal one for any unit that is not an island; that is, for any unit that is not hypothesized to be spatially independent from all other units of analysis and does not have zeroes across an entire row of $\mathbf{W}$. For samples with islands, $\boldsymbol{W\iota}$ will be equivalent to a dummy variable in which a 0 indicates that a unit is an island. For samples without islands, however, $\boldsymbol{W\iota}$ will result in a vector of ones. Additionally, specifications of $\mathbf{W}$ using the k nearest-neighbor will also result in $\boldsymbol{W\iota}$ equaling a constant, whether they are row-standardized or not. So long as a constant is already estimated within the model, then scholars do not need to worry about estimating $\boldsymbol{W\iota}$ in these niche circumstances.

Additionally, many scholars are using spatial regression models with panel data. Usually, this is done by taking the Kronecker product of a cross-sectional $\mathbf{W}$ matrix and an identity matrix. This will result in $\mathbf{W}$ only capturing cross-sectional dependence and, as a result, $\boldsymbol{W\iota}$ will only have cross-sectional variation. Thus, $\boldsymbol{W\iota}$ cannot be estimated alongside cross-sectional fixed effects, though any bias resulting from omitting $\boldsymbol{W\iota}$ will be captured by these fixed effects. This mirrors how the other set of constitutive terms $\mathbf{Kx}$ cannot be estimated alongside temporal fixed effects (Vande Kamp n.d.).

4 Interpretational Gains from a Multiplicative Interaction Framework

Thus far, the multiplicative interaction frame of spatial regression models has highlighted the need to include $\boldsymbol{W\iota}$ as a variable. This was motivated from endogeneity concerns. But the multiplicative interactions framework also allows new possibilities for interpreting spatial regression models. At present, the interpretation of spatial regression models focuses on the direct and indirect effects of a given independent variable $\mathbf{x}$. A non-spatial regression model assumes that a given unit's observation x_i only has a direct effect on that unit's dependent variable y_i. Spatial regression models allow the possibility, however, that x_i will have an indirect effect on other units y_j, which is only possible through the creation of spatial lags. A summary of these direct and indirect effects are given in Table 1.

Table 1: Direct and Indirect Effects of a Variable Using Different Spatial Regression Model Specifications

Model	Direct Effect	Indirect Effect
OLS/SEM	β	0
SLX	β	θ
SAR	Diagonal elements of $(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\beta$	Non-diagonal elements of $(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\beta$

Reproduced from Halleck Vega and Elhorst (2015). Specifications involving permutations of these models are omitted for simplicity.

In a SLX model, interpretation is relatively straightforward: the direct effect of $\mathbf{x}$ on $\mathbf{y}$ is captured by its coefficient, β, while the indirect effect is captured by the coefficient of the spatial lag of $\mathbf{x}$, θ. But while this separation of effects is useful for hypothesis testing, direct interpretation of the SLX coefficient is difficult. A one-unit increase in a spatial lag of $\mathbf{x}$ can result from a change in one or more x_j, one or more w_{ij}, or a combination of the two. This contrasts with the standard regression context, in which a one-unit increase in an independent variable often has a clean substantive interpretation.

In a SAR model, interpretation becomes considerably more complicated. This complication arises from the nature of SAR models, which allow for global spillovers of effects. Suppose a scholar wanted to interpret the following (restricted) SAR model:

$$\boldsymbol{y} = \boldsymbol{\iota}\alpha + \boldsymbol{x}\beta + \boldsymbol{W}\boldsymbol{y}\boldsymbol{\rho} + \boldsymbol{\varepsilon} \tag{17}$$

Because $\mathbf{Wy}$ necessarily contains $\mathbf{Wx}$, the effect of the independent variable $\mathbf{x}$ on $\mathbf{y}$ is unclear. Thus scholars use this transform this equation to isolate $\mathbf{y}$ on the right hand side which, in turn, produces the core matrix of interest from Table 1:

$$\boldsymbol{y} - \boldsymbol{W}\boldsymbol{y}\boldsymbol{\rho} = \boldsymbol{\iota}\alpha + \boldsymbol{x}\beta + \boldsymbol{\varepsilon} \tag{18}$$

$$(\boldsymbol{I} - \rho\boldsymbol{W})\boldsymbol{y} = \boldsymbol{\iota}\alpha + \boldsymbol{x}\beta + \boldsymbol{\varepsilon} \tag{19}$$

$$\boldsymbol{y} = (\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\boldsymbol{\iota}\alpha + (\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\boldsymbol{x}\beta + (\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\boldsymbol{\varepsilon} \tag{20}$$

The matrix $(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}$ is called the spatial multiplier matrix. This matrix contains an infinitely repeating series of $\mathbf{W}$ matrices of the following form:

$$(\boldsymbol{I} - \rho\boldsymbol{W})^{-1} = \boldsymbol{I} + \rho\boldsymbol{W} + \rho^2\boldsymbol{W}^2 + \rho^3\boldsymbol{W}^3 + ... \tag{21}$$

This expansion demonstrates the difficulties with interpreting SAR models. $\mathbf{I}$ is an identity matrix and, because it has zeroes along the non-diagonal elements, only returns the direct effect of x_i on y_i when post-multiplied by $\boldsymbol{x}\beta$. Similarly, when $\rho\boldsymbol{W}$ is post-multiplied by $\mathbf{x}$, it only returns the indirect effects of x_j on y_i because $\mathbf{W}$ has zeroes along the diagonal. The higher order $\mathbf{W}$ matrices, however, contain both direct and indirect effects due to the feedback present spatial terms (Elhorst 2014). For the second order $\mathbf{W}$ matrix, for example, unit i will affect its neighbors, including unit j, which will then affect unit j's neighbors, which also include unit i. This feedback makes interpretation of direct and indirect effects in SAR models extremely difficult and, as a result, each unit has its own unique direct effect as well as a unique indirect effect on every other unit in the sample. Of course, $(\boldsymbol{I} - \rho\boldsymbol{W})^{-1}\boldsymbol{x}\beta$ contains an extraordinarily large amount of information; this has lead many scholars to call for the presentation of average direct and indirect effect estimates (LeSage and Pace 2009). Standard errors for all of these marginal effects have to be computed through simulation.

These difficulties in interpreting spatial regression models are unlikely to go away, particularly for the SAR specification. But the multiplicative interactions framework for these models also opens up additional interpretive possibilities that may be attractive to scholars. Recall that while spatial regression models contain n-1 multiplicative interactions, they are linearly restricted to have the same coefficient(s). Thus, one can find the marginal effect an increase in any w_{ij} on y_i in an SLX model, for example, by taking the partial derivative of equation 9 with respect to w_{ij}:

$$\frac{\partial y_i}{\partial w_{ij}} = \theta_2 + \theta_3 x_j \tag{22}$$

This conditional marginal effect is relatively straightforward to interpret using marginal effects plots (Brambor, Clark, and Golder 2006). Indeed, such a plot would concisely display a marginal effect that is generalizable to a change in the spatial relationship between any two units in the data. While certainly useful in the SLX context, such a presentation is a notable simplification over current SAR interpretations using the spatial multiplier. A marginal effect plot avoids the need for presenting either the enormous amount of information in $(\boldsymbol{I}-\rho\boldsymbol{W})^{-1}\boldsymbol{x}\beta$ or the overly simplified information contained in averages of direct and indirect effects. Indeed, this approach also includes a straightforward way to calculate standard errors for marginal effects that avoids the need for simulation. The marginal effect of x_j on y_i can also be presented using marginal effects plots. By doing so, it allows for a straightforward, substantive interpretation of the coefficients of spatial lags that is not currently possible, as described above.

Of course, using marginal effect plots will not totally simplify the interpretation of spatial regression models. In SAR models, the use of marginal effect plots will be necessarily restricted to the effect of w_{ij} and y_j on y_i. For other variables in the model, the use of the spatial multiplier will still be necessary in order to draw substantive conclusions about the direct and indirect effects of a variable. The advances described here are a complement, not a substitute, for current practices in the literature.

5 Monte Carlo Design

Two interrelated arguments have been advanced concerning the estimation of spatial regression models. First, the information in $\mathbf{W}$ may be predict important phenomena outside of their role in creating spatial lags and warrants the inclusion of $\boldsymbol{W\iota}$ as an independent variable in spatial regression models. Second, omitting $\boldsymbol{W\iota}$ when it is part of the data-generating process of $\mathbf{y}$ will lead to biased coefficients of spatial lags and hamper inferences. If these two arguments are true, then spatial regression models are inherently multiplicative interaction models, a framework which allows for new methods of interpreting spatial regression models by deriving conditional marginal effects and plotting them.

The validation of these two different points requires two different approaches. The question of whether $\boldsymbol{W\iota}$ predicts important social phenomena is an empirical one. To that end, multiple empirical applications are presented towards the end of the paper to demonstrate the necessity of this multiplicative interactions framework within social science. In contrast, the question of whether omitting $\boldsymbol{W\iota}$ when it is part of the data-generating process will lead to biased coefficient estimates is an analytic one. To demonstrate that this is true, I rely on Monte Carlo simulations.

The design of the Monte Carlo analysis will be crucial in demonstrating my point. First, the Monte Carlo design must be a SLX model. Spatial lags of the dependent variable are necessarily correlated with all of the other independent variables in the model. Demonstrating that estimates of the coefficient of spatially lagged dependent variables are biased in the absence of $\boldsymbol{W\iota}$ does little to prove my example. Thus, I must demonstrate with an SLX model that omitting $\boldsymbol{W\iota}$ can lead to biased estimates of the coefficient of spatially lagged independent variables.

The design of the Monte Carlo analysis also must demonstrate that all spatial regression models are multiplicative interaction models. This stands in contrast to the approach of Neumayer and Plumper (2012), who argue spatial regression models that do not row-standardize $\mathbf{W}$ are multiplicative interaction models (and no further). The Monte Carlo analysis presented here do not use row-standardized matrices, as that is no longer common practice. Additional analysis of row-standardized matrices is presented in the supplemental materials; the results show that (TBD).

I present two sets of analysis to demonstrate that spatial regression models are multiplicative interaction models. First, I demonstrate that the row-sum of $\mathbf{W}$, $\boldsymbol{W\iota}$, will necessarily be correlated with a spatially lagged variable $\mathbf{Wx}$, where $\mathbf{x}$ is not a spatially autoregressive variable. I generate one thousand observations of $\mathbf{x}$ from a uniform distribution U (and, after generating $\mathbf{W}$, calculate both $\mathbf{Wx}$ and $\boldsymbol{W\iota}$. I then calculate the correlation between the two variables and its 95% confidence interval, recording each. This process is repeated one thousand times, with the average correlation and the proportion of correlations statistically distinguishable from zero presented in the paper. If spatial regression models are multiplicative interaction models, then the correlation between $\mathbf{Wx}$ and $\boldsymbol{W\iota}$ should be statistically distinguishable from zero at a rate greater than chance.

I generate a number of different $\mathbf{W}$ specifications in order to determine . Most specifications rely on each observation of $\mathbf{x}$ being randomly located in a bounded two-dimensional space, with $\mathbf{W}$ being calculated based on the Euclidean distance between the two variables. Matrices based on linear distance, inverse distance, a small sphere of influence, and a large sphere of influence are created. Additionally, non-symmetric $\mathbf{W}$ matrices are generated by randomly generating the non-diagonal elements of $\mathbf{W}$ from a uniform distribution. I compute two classes of non-symmetric $\mathbf{W}$: those with only positive values and those with positive and negative values. For all specification types, a new $\mathbf{W}$ is generated for each set correlation calculated. The Monte Carlo results, then, are not dependent on a few idiosyncratic specifications of $\mathbf{W}$.

Second, I demonstrate that omitting $\boldsymbol{W\iota}$ when it is part of the data generating process of a dependent variable will result in biased estimates of spatially lagged independent variables. One thousand data points are generated following this SLX data generating process:

$$\boldsymbol{y} = \alpha + \boldsymbol{x}\beta + \boldsymbol{W\iota}\theta_1 + \boldsymbol{W x}\theta_3 + \boldsymbol{\varepsilon}$$

where $\alpha = 10$, $\beta = 5$ and $\theta_3 = 3$. $\boldsymbol{\varepsilon}$ is drawn from a standard normal distribution, while $\boldsymbol{x}$ is generated from a uniform distribution, U (-1, 1). $\mathbf{W}$ is randomly generated according to either a small sphere of influence specification mentioned or a non-symmetric $\mathbf{W}$ with positive and negative non-diagonal elements, resulting in two distinct sets of analysis depending on the class of $\mathbf{W}$ employed.

An OLS model is then estimated on the data that that includes $\mathbf{x}$ and $\mathbf{Wx}$, but not

$\boldsymbol{W\iota}$. Coefficients and standard errors are recorded, and this process is repeated one thousand times. θ_1 varies over the simulations in order to demonstrate the biases resulting from omitting the variable; when θ_1 equals zero, this reduces to a correctly specified model. I subsequently report the average bias for all coefficient estimates and their coverage probabilities using 95% confidence intervals. If spatial regression models are multiplicative interaction models, then estimates of θ_3 should be unbiased when the model is correctly specified but biased and have poor coverage when a relevant $\boldsymbol{W\iota}$ is omitted (a coverage probability less than 0.95).

6 Monte Carlo Analysis

The results of the correlation simulations are presented in Table 2. The average correlation between $\mathbf{Wx}$ and $\boldsymbol{W\iota}$ is small, likely due to the face that both $\mathbf{x}$ and $\mathbf{W}$ are randomly generated for each set of observations. But importantly for my argument, the proportion of cases statistically distinguishable from zero are high. Every time a symmetric $\mathbf{W}$ is used, the proportion is between 70% and 90%. This provides strong evidence that the two quantities are correlated. For non-symmetric matrices, the proportion of statistically significant cases is much lower: about 20% for each one. Still, this rate is four times greater than we expect to observe from chance alone. Even in these general cases, there is reason to suspect that $\mathbf{Wx}$ and $\boldsymbol{W\iota}$ will be correlated. This, in turn, provides evidence for my argument that spatial regression models are multiplicative interaction models.

The results of the regression simulations are presented in Tables 3 and 4. The results of the simulations with a small sphere of influence specification for $\mathbf{W}$ are in Table 3. They clearly indicate that omitting a relevant $\boldsymbol{W\iota}$ biases estimates of spatial regression models; only when $\theta_1 = 0$ are the estimates unbiased and coverage probabilities correct. By far and away, the bias when a relevant $\boldsymbol{W\iota}$ is omitted is concentrated on estimates of α and θ_3.

Table 2: Average Correlation Between $\boldsymbol{W x}$ and $\boldsymbol{W \iota}$ (Proportion of Statistically Significant Correlations in Parentheses)

$\boldsymbol{W}$	Average Correlation	Proportion Statistically Significant
Linear Distance	-0.01	0.90
Inverse Distance	0.00	0.71
Small Sphere of Influence	0.00	0.84
Large Sphere of Influence	0.01	0.89
Random **W**, + Values	-0.00	0.19
Random **W**, +/- Values	0.00	0.18

Both quantities are heavily biased and rarely include the population parameter in confidence intervals; the intercept estimate never includes the true population value in any model. Estimates of β, while containing better coverage probabilities than other estimates, also fall short of 0.95. The bias in α follows a predictable direction: negative when θ_2 is negative, positive when positive. In contrast, the bias in β and θ_3 do not follow any appreciable pattern.

Table 3: Bias in SLX Models using a small sphere of influence specification of **W** when $\boldsymbol{W \iota}$ is omitted (Coverage Probabilities in Parentheses)

	Population Value of θ_1				
Coefficient	-5	-2.5	0	2.5	5
α	-1041.06	-520.02	0.00	520.99	1044.14
	(0.00)	(0.00)	(0.94)	(0.00)	(0.00)
β	0.10	0.40	-0.00	-0.29	0.26
	(0.87)	(0.86)	(0.94)	(0.88)	(0.88)
θ_3	-0.07	0.19	-0.00	-0.20	0.12
	(0.16)	(0.12)	(0.94)	(0.14)	(0.14)

The results of the simulations using a non-symmetric **W** with random positive and negative non-diagonal elements are in Table 4. In contrast to the previous example, the problems from omitting $\boldsymbol{W \iota}$ are much less severe. The estimates of α and β are biased very little on average and properly cover the true population parameter. The estimates of θ_3 are also biased very little on average. While the coverage probability is lower than 0.95 when a substantive $\boldsymbol{W \iota}$ is omitted, that coverage probability does not drop below 0.8. This indicates that the non-symmetric nature of this matrix greatly helps in getting unbiased estimates. Still, this model assumes there is no measurement error in $\mathbf{x}$. Should measurement error occur in $\mathbf{x}$, specifically when the variable is inflated by a constant, there will likely be more extreme bias permeate the model.

Table 4: Bias in SLX Models using a non-symmetric $\mathbf{W}$ with random positive and negative non-diagonal elements when $\boldsymbol{W\iota}$ is omitted (Coverage Probabilities in Parentheses)

	Population Value of θ_2				
Coefficient	-5	-2.5	0	2.5	5
α	0.03	0.03	-0.00	-0.04	0.10
	(0.94)	(0.95)	(0.96)	(0.95)	(0.94)
β	-0.01	-0.02	0.00	-0.06	0.17
	(0.95)	(0.95)	(0.95)	(0.96)	(0.95)
θ_1	0.01	0.00	0.00	-0.00	0.01
	(0.81)	(0.85)	(0.94)	(0.85)	(0.83)

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